A RESULT ON CLASS-$\mathcal{C}^1$ LINEARIZATION OF CONTRACTIONS IN INFINITE DIMENSIONS.

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1. Motivation.

This note is a short presentation of our results in [5]. We start explaining, as a motivating example, a situation where a result of $\mathcal{C}^1$-linearization in infinite dimensions was needed and used.

In the paper [2] it was proved that for some nonlinearities $f(x,u)$ and some small values of $\alpha > 0$ the global attractor of the dynamical system defined in $H^1(0,\pi) \times L^2(0,\pi)$ by the second order initial-boundary value problem

$$\begin{aligned} &u_{tt} + 2\alpha u_t = u_{xx} + f(x,u), \quad 0 < x < \pi \\ &u_x(0) = u_x(\pi) = 0 \end{aligned}$$

is not contained on any finite-dimensional invariant manifold of class $\mathcal{C}^1$.

In one of the steps of the proof it was needed to prove that for the case $f(x,u) \equiv f(u)$ with $f(0) = 0$ and $f'(0) < 0$ there are only countable many finite-dimensional invariant manifolds of class $\mathcal{C}^1$ containing the (assimptotically stable) equilibrium point $(u, u_t) = (0,0)$.

This fact was proved by linearization, that is by showing that under an abstract change of variables of class $\mathcal{C}^1$ in a neighborhood of $(0,0) \in H^1(0,\pi) \times L^2(0,\pi)$, the equation turned into its linear part

$$\begin{aligned} &v_{tt} + 2\alpha v_t = v_{xx} + f'(0)v, \quad 0 < x < \pi \\ &v_x(0) = v_x(\pi) = 0. \end{aligned}$$

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This last equation can be analyzed with detail because it is easily solved with the method of separation of variables:

$$u(x,t) = \sum_{n=-\infty}^{n=+\infty} c_n e^{t\lambda_n} \cos nx$$

with

$$\lambda_{\pm n} = -\alpha \pm \sqrt{\alpha^2 + 4f'(0) - 4n^2}.$$

We observe that if $\alpha^2 < 4|f'(0)|$ then all the eigenvalues are simple, nonreal and all of them have the same real part: $\mathrm{Re}(\lambda_n) = -\alpha \ \ (< 0)$.

The fact that this linear equation has only countable many finite-dimensional invariant manifolds of class $\mathcal{C}^1$ containing the point $(v, v_t) = (0,0)$ is not completely easy to see. The proof uses strongly the fairly non-generic fact that all the eigenvalues are different, but with the same real part. It turns out that these manifolds are in fact the linear subspaces generated by a finite number of pairs of conjugated eigenfunctions.

In any case, it appears as something very natural the need of a linearization result to relate this property for the linear and the nonlinear equations, and also the need of this linearization to be of class $\mathcal{C}^1$, in order to preserve the $\mathcal{C}^1$ manifolds. In particular, it is clear that the well-known result of Lipschitz linearization due to Ch. Pugh [3], valid in infinite-dimensional spaces (even with additional symmetry restrictions, as it is shown in [4]), is not enough to deal with this situation.

2. Previous linearization results.

The precise statement proved in that paper is the following, that deals with the more general case of maps, instead of flows:

Theorem ([2]):

Let $\mathbb{X}$ be a Banach space, $T = A + \mathcal{X} : \mathbb{X} \to \mathbb{X}$ such that $A, A^{-1} \in \mathcal{L}(\mathbb{X})$ and $\mathcal{X} \in \mathcal{C}^1(\mathbb{X}, \mathbb{X})$ and such that $\mathcal{X}(0) = 0$. Assume that there exists an $\eta > 0$ with the following properties

$$\|A^{-1}\| \|A\|^{1+\eta} < 1 \tag{2.1}$$

$$D\mathcal{X}(x) = o(\|x\|^{\eta}) \quad \textit{as } x \to 0. \tag{2.2}$$

Then, there exists a local map $\phi \in \mathcal{C}^1(X, X)$ with $\phi(0) = 0$ and $D\phi(x) = o(\|x\|^\eta)$ as $x \to 0$ such that, if we define $R = I + \phi$ then we have

$$RT = AR$$

(or $RTR^{-1} = A$) in a neighborhood of zero.

Observe that inequality (2.1) implies that A is a contraction. The existence of an equivalent norm in $\mathbb{X}$ where (2.1) holds is equivalent to the existence of numbers $0 < \mu < \nu < 1$ such that the spectrum of A, $\sigma(A)$ satisfies $\sigma(A) \subset \{\mu < |\lambda| < \nu\}$ and that the "non-resonance" condition $\nu^{1+\eta} < \mu$ holds.

Condition (2.2) holds automatically for every $\eta < 1$ if $\mathcal{X}$ is of class $\mathcal{C}^2$. It is like a Lipschitz-Hölder condition only at $x = 0$. Unlike the global global Lipschitz-Hölder conditions it is also meaningful if $\eta > 1$, though quite non-generic.

We give also an idea of the proof, to see where the non-resonance condition appears:

We write $R = A^{-1}RA$, or

$$\phi(x) = A^{-1}\phi(Ax + \mathcal{X}(x)) + A^{-1}\mathcal{X}(x),$$

a linear non-homogeneous equation in the unknown ϕ. The right hand side will be a contraction in the norm

$$\|\phi\|_\eta = \sup_{0<\|x\|<\delta} \|x\|^{-\eta}\|D\phi(x)\|$$

for a suitable $\delta > 0$.

Let us see that it in the (easiest) case $\mathcal{X} \equiv 0$ the transformation $\phi(x) \mapsto A^{-1}\phi(Ax)$ is actually a contraction:

$$\|x\|^{-\eta}\|A^{-1}D\phi(Ax)A\| \leq$$
$$\|A^{-1}\|\|A\|\frac{\|D\phi(Ax)\|}{\|Ax\|^\eta}\frac{\|Ax\|^\eta}{\|x\|^\eta} \leq \|A^{-1}\|\|A\|^{1+\eta}\|\phi\|_\eta.$$

And this is the point where the condition (2.1) appears.

The following older result, perhaps not very well known, showed that the $\mathcal{C}^1$ linearization of a contraction is always possible in finite dimensions, without non-resonance additional conditions:

Theorem (P. Hartman, [1]):

Let A be an $n \times n$ invertible matrix such that $\|A\| < 1$ and $\mathcal{X} \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$ be such that $\mathcal{X}(0) = 0$ and $D\mathcal{X}(0) = 0$. Then, for the map $Tx = Ax + \mathcal{X}(x)$ there exists a map $Rx = x + \phi(x)$ in a neighborhood of zero with $\phi \in \mathcal{C}^{1+\eta'}$ (with $0 < \eta'$), $\phi(0) = 0$ and $D\phi(0) = 0$ such that $RTR^{-1} = A$.

Hartman's proof is an induction process on the moduli of the eigenvalues of A. So, at a first sight, could seem hard to extend to the infinite-dimensional case.

3. Our result.

The following is our main result:

Theorem([5]):

Let $\mathbb{X}$ be a Banach space such that there exists a $C^{1,1}$-function $\gamma : \mathbb{X} \to \mathbb{R}$, with $\gamma(x) = 1,\ |x| \leq 1/2,\ \gamma(x) = 0,\ |x| \geq 1$. Assume that $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n$.
Suppose that $A,\ A^{-1} \in \mathcal{L}(\mathbb{X})$ and that $A =$ diag $(A_1, A_2, \cdots, A_n)$, where $A_i \in \mathcal{L}(\mathbb{X}_i),\ i = 1, \cdots, n$.
Let $\mu_i,\ \nu_i,\ i = 1, \cdots, n$ be such that

$$0 < \mu_n < \nu_n < \mu_{n-1} < \nu_{n-1} < \cdots \mu_1 < \nu_1 < 1$$
$$\nu_1 \nu_i < \mu_i,\ i = 1, \cdots, n$$
$$|A_i| < \nu_i,\ |A_i^{-1}| < 1/\mu_i \quad i = 1, \cdots, n$$

Let $\mathcal{X} = \mathcal{X}(x)$ be a $C^{1,1}$-function in a neighborhood of the origin, such that $\mathcal{X} = 0,\ \partial_x \mathcal{X} = 0$ at $x = 0$.

Then, for the map $Tx = Ax + \mathcal{X}(x)$ there exists a C^1-map $Rx = x + \phi(x)$ satisfying $\phi = 0,\ \partial_x \phi = 0$ at $x = 0$, such that $RTR^{-1} = A$ in a sufficiently small neighborhood of the origin.

Comments:

i) The Banach spaces that fulfill the first condition are said to be spaces of class $\mathcal{C}^{1,1}$. Hilbert spaces, for example, have this property.

ii) If the set of spectral radii of A, $\sigma r(A) = \{|\lambda| \textit{ such that } \lambda \in \sigma(A)\}$ has empty interior, for example, one can always make a decomposition of the space into a finite number of invariant subspaces satisfying this non-resonance condition.

iii) Because of ii), our theorem includes Hartman's. It also includes the main cases of [2] (except regularity precisions) as the case $n = 1$.

iv) The proof follows the steps of Hartman's, but considering spectral sets (or blocks) $\mu_i < |\lambda| < \nu_i$ instead of moduli of eigenvalues. We also make more systematic use of fixed points of contractions.

The proof is by induction on the number n of blocks. The typical induction step can be represented by the case $n = 3$ as:

With $x = (x_1, x_2, x_3)$, suppose that

$$T(x) = (A_1x_1, A_2x_2 + \mathcal{X}_2(x), A_3x_3 + \mathcal{X}_3(x)).$$

We want to perform a $\mathcal{C}^1$ change of variables R such that

$$RTR^{-1}(x) = (A_1x_1, A_2x_2, A_3x_3 + \mathcal{X}_3'(x)).$$

To do that, we see first that after a preliminary change of variables we can suppose that $\mathcal{X}_2(x_1, 0, 0) \equiv 0$, $\mathcal{X}_3(x_1, 0, 0)) \equiv 0$. This amounts as to obtain an invariant manifold tangent to $\mathbb{X}_1$, of the form $x_2 = x_2(x_1)$, $x_3 = x_3(x_1)$. It is not difficult to convince oneself that with this change of variables the regularity with respect to the variable x_1 can be reduced to be merely of class $\mathcal{C}^{1,\beta}$, for some small $\beta > 0$.

Then, as a second step, we need a solution $\phi : \mathbb{X} \to \mathbb{X}_2$ with $\phi(x_1, 0, 0) \equiv 0$, $\partial_{(x_2,x_3)}\phi = 0$, at $(0, 0, 0)$ of the functional equation $\phi(x) = A_2^{-1}\phi(A_1x_1, A_2x_2 + \mathcal{X}_2(x), A_3x_3 + \mathcal{X}_3(x)) - A_2^{-1}\mathcal{X}_2(x)$.

This is expected to be seen as a fixed point of a contraction, at least in the auxiliary semi-norm

$$\|\phi\|_{aux} := \sup \frac{|\partial_{x_1}\phi(x_1, x_2, x_3) - \partial_{x_1}\phi(0, x_2, x_3)|}{|x_1|^\beta(|x_2| + |x_3|)^\eta},$$

for some $\beta + \eta < 1$.

In the simplest case $\mathcal{X}_2 \equiv 0$ and $\mathcal{X}_3 \equiv 0$ the operator we want to be a contraction is $K[\phi](x) := A_2^{-1}\phi(A_1x_1, A_2x_2, A_3x_3)$. In this case, we see the appearance of the non resonance condition, since one easily obtains that

$$\|K\,[\phi]\,\|_{aux} \leq \|A_1\|^{1+\beta}\|A_2^{-1}\|\|A_2\|^{\eta}\|\phi\|_{aux},$$

so it is a contraction if we choose (β, η) near $(0, 1)$.

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